The multi-variable spectral geometric mean

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Abstract

In the 1980s, Kubo and Ando introduced the operator mean of positive operators. One of the important of operator means is the weighted metric geometric mean,

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \ A, B \in \mathbb{P},$$

where $\mathbb{P}$ denotes the convex cone of all invertible positive definite operators. A natural extension of this is the **multi-variable metric geometric mean**, defined as the least squares mean with respect to the Riemannian distance $d_R(A, B) := \| \log A^{-1/2}BA^{-1/2}\|_2$. Another important mean is the **spectral geometric mean** of two positive definite matrices, introduced by Fiedler and Pták, which is given by

$$A\natural_t B = (A^{-1}\#_{1/2}B)^t A(A^{-1}\#_{1/2}B)^t = (I\#_t A^{-1}\#_{1/2}B)A(I\#_t A^{-1}\#B).$$

We define the **multi-variable spectral geometric mean** and study its fundamental properties. Also, we discuss another least squares mean with respect to the Wasserstein distance
$d_W(A, B) = \left(\operatorname{tr}(A + B) - 2\operatorname{tr}\left((A^{1/2}BA^{1/2})^{1/2}\right)\right)^{1/2}$, which is known as the Wasserstein mean. Finally, we discuss gradient descent algorithms for the Wasserstein mean.